Solution 6.1

(a) The Hamiltonian for a particle of mass m in a one-dimensional harmonic oscill
potential is

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$$

where the particle moves in the potential $V(x) = \kappa x^2$ and oscillates at freque
$\omega = \sqrt{\kappa/m}$, where κ is a force constant. The operator $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$.

(b) The Hamiltonian can be factored

$$\hat{H} = \frac{m\omega^2}{2}\left(\frac{\hat{p}_x^2}{m^2\omega^2} + \hat{x}^2\right)$$

so it makes sense to define new operators

$$\hat{b} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i\hat{p}_x}{m\omega}\right)$$

$$\hat{b}^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i\hat{p}_x}{m\omega}\right)$$

which, can be written in terms of operators $\hat{x}$ and $\hat{p}_x$ to give

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{b} + \hat{b}^\dagger)$$

$$\hat{p}_x = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\hat{b}^\dagger - \hat{b})$$

Substituting this into the Hamiltonian gives

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \frac{-1}{2m}\left(\frac{\hbar m\omega}{2}\right)(\hat{b}^\dagger - \hat{b})^2 + \frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}\right)(\hat{b}^\dagger + \hat{b})^2$$

$$\hat{H} = \left(\frac{\hbar\omega}{4}\right)(-\hat{b}\hat{b} - \hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}) + \left(\frac{\hbar\omega}{4}\right)(\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b})$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b})$$

(c) To derive the commutation relation $[\hat{b}, \hat{b}^\dagger] = \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}$ we can write out
differential form of the operators and have them act on a dummy wave function. T
gives

$$(\hat{b}\hat{b}^\dagger)\psi = \left(\frac{m\omega}{2\hbar}\right)\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi$$

$$(\hat{b}\hat{b}^\dagger)\psi = \left(\frac{m\omega}{2\hbar}\right)\left(x^2 + \frac{\hbar}{m\omega} + \frac{\hbar}{m\omega}x\frac{\partial}{\partial x} - \frac{\hbar}{m\omega}x\frac{\partial}{\partial x} - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right)\psi$$

and

$$(\hat{b}^\dagger\hat{b})\psi = \left(\frac{m\omega}{2\hbar}\right)\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi$$

$$(\hat{b}^\dagger\hat{b})\psi = \left(\frac{m\omega}{2\hbar}\right)\left(x^2 - \frac{\hbar}{m\omega} - \frac{\hbar}{m\omega}x\frac{\partial}{\partial x} + \frac{\hbar}{m\omega}x\frac{\partial}{\partial x} - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\right)\psi$$

so that the commutation relation simplifies to just

$$(\hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b})\psi = \left(\frac{m\omega}{2\hbar}\right)\left(\frac{2\hbar}{m\omega} - \frac{2\hbar}{m\omega}x\frac{\partial}{\partial x} + \frac{2\hbar}{m\omega}x\frac{\partial}{\partial x}\right)\psi = \psi$$

or, in even more compact form,

$$[\hat{b}, \hat{b}^\dagger] = \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = 1$$

If we do not want to use differential operators and a dummy wave function then we could write

$$[\hat{b}, \hat{b}^\dagger] = \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x} + \frac{i\hbar\hat{p}_x}{m\omega}\right)\left(\hat{x} - \frac{i\hbar\hat{p}_x}{m\omega}\right) - \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x} - \frac{i\hbar\hat{p}_x}{m\omega}\right)\left(\hat{x} + \frac{i\hbar\hat{p}_x}{m\omega}\right)$$

$$[\hat{b}, \hat{b}^\dagger] = \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x}^2 + \frac{i\hbar\hat{p}_x\hat{x}}{m\omega} - \frac{i\hbar\hat{x}\hat{p}_x}{m\omega} + \frac{\hbar^2\hat{p}_x^2}{m^2\omega^2}\right) - \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x}^2 - \frac{i\hbar\hat{p}_x\hat{x}}{m\omega} + \frac{i\hbar\hat{x}\hat{p}_x}{m\omega} + \frac{\hbar^2\hat{p}_x^2}{m^2\omega^2}\right)$$

$$[\hat{b}, \hat{b}^\dagger] = \left(\frac{im\omega}{\hbar}\right)\left(\frac{\hbar\hat{p}_x\hat{x}}{m\omega} - \frac{\hbar\hat{x}\hat{p}_x}{m\omega}\right) = i(\hat{p}_x\hat{x} - \hat{x}\hat{p}_x) = i[\hat{p}_x, \hat{x}] = i(-i\hbar) = 1$$

since $[\hat{p}_x, \hat{x}] = -i\hbar$.

(d) The Hamiltonian is

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}) = \frac{\hbar\omega}{2}(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} + \hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}) = \frac{\hbar\omega}{2}(1 + 2\hat{b}^\dagger\hat{b}) = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$$

where we made use of the commutation relation $[\hat{b}, \hat{b}^\dagger] = \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = 1$

Solution 6.2

(a) The expectation value of position $\langle x\rangle$ and momentum $\langle p\rangle$ for the first excited state $|n = 1\rangle$ of a particle of mass m in a one-dimensional harmonic oscillator potential is found using

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{b} + \hat{b}^\dagger)$$

$$\hat{p}_x = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\hat{b}^\dagger - \hat{b})$$

and

$$|\hat{b}^\dagger n\rangle = (n+1)^{1/2}|n+1\rangle$$

$$|\hat{b}n\rangle = n^{1/2}|n-1\rangle$$

$$\langle m|n\rangle = \delta_{mn}$$

so that

$$\langle x\rangle = \langle 1|x|1\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\langle 1|\hat{b} + \hat{b}^\dagger|1\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\langle 1|0\rangle + \sqrt{2}\langle 1|2\rangle) = 0$$

and

$$\langle p_x\rangle = \langle 1|\hat{p}_x|1\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}\langle 1|(\hat{b}^\dagger - \hat{b})|1\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\sqrt{2}\langle 1|2\rangle - \langle 1|0\rangle) = 0$$

(b) To find the value of the product in uncertainty in position Δx and momentum Δp_x for the first exited state of a particle of mass m in a one-dimensional harmonic oscillator potential we use

$\Delta x = (\langle x^2\rangle - \langle x\rangle^2)^{1/2}$

and

$\Delta p_x = (\langle p^2{}_x\rangle - \langle p_x\rangle^2)^{1/2}$

and since $\langle x\rangle = 0$ and $\langle p_x\rangle = 0$ we will be interested in finding the value of $\langle x^2\rangle$ and $\langle p^2{}_x\rangle$. Starting with $\langle x^2\rangle$, we have

$$\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)\langle 1|(\hat{b}+\hat{b}^\dagger)^2|1\rangle = \left(\frac{\hbar}{2m\omega}\right)\langle 1|\hat{b}\hat{b}+\hat{b}^\dagger\hat{b}^\dagger+\hat{b}\hat{b}^\dagger+\hat{b}^\dagger\hat{b}|1\rangle = \frac{3\hbar}{2m\omega}$$

here we used the fact that $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{b}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. One can see that for the general state $|n\rangle$ one has $\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)(1+2n)$. Now turning our attention to $\langle p^2{}_x\rangle$ we have

$$\langle p^2{}_x\rangle = -\left(\frac{\hbar m\omega}{2}\right)\langle 1|(\hat{b}^\dagger-\hat{b})^2|1\rangle = \left(\frac{\hbar m\omega}{2}\right)\langle 1|-\hat{b}\hat{b}-\hat{b}^\dagger\hat{b}^\dagger+\hat{b}\hat{b}^\dagger+\hat{b}^\dagger\hat{b}|1\rangle = \frac{3\hbar m\omega}{2}$$

and one can see that for the general state $|n\rangle$ one has $\langle \hat{p}_x^2\rangle = \left(\frac{\hbar m\omega}{2}\right)(1+2n)$.

For the particular case we are interested in $|n=1\rangle$ and the uncertainty product is

$$\Delta x\Delta p_x = (\langle x^2\rangle\langle p^2{}_x\rangle)^{1/2} = \frac{3}{2}\hbar$$

For the general state $|n\rangle$ the uncertainty product $\Delta x\Delta p_x = \frac{\hbar}{2}(1+2n)$